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On Newton's paradoxes

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Abstract
Reading the **Principia** [1] it is easy to realize that Newton, without being aware of the fact, has corrected the law of the fall of bodies formulated by Galilei. In the present paper we examine the consequences of such a fact and show that they affect the entire Celestial Mechanics, being related to the very small 43" per century of the perihelion shift of Mercury as well as to the substantial 50" per year of the grand phenomenon of the lunisolar precession.

key word: Galileo – Newton – Le Verrier – Einstein – General Relativity – Perihelion of Mercury – Law Hall's law – Precession.

1 Introduction

The greatest scientific work of all times, **Philosophiae Naturalis Principia Mathematica** [1], by Isaac Newton, was published for the first time in July 1687, in a few copies. In the following three centuries it got some criticism, but in practice it remained unequalledly unaltered, since the nature of all the great contributions produced in that age is exclusively mathematical. But one of the few points that became subjects of discussions was the distinction between absolute and relative motions. Actually Newton states that when a motion gives rise to inertial forces then it is absolute. On this subject, in the Scolio IV of Definition VIII, he describes the well-known example of the bucket of water, hanged by means of a long thread and rotating. It is known that the water, after some time, tends to rise towards the edge of the container and so Newton deduces that this is a case of absolute motion. The archbishop Berkeley expressed its opinion against this conclusion. His reasoning was that if all the universe rotated with opposite speed around the bucket, and this was still, the same phenomenon would occur. In more recent times, E. Mach [2] agreed with the criticism of Berkeley and in this connection textually stated: *Try to hold still the Newtonian bucket, and to rotate the sky of the stars and to verify the absence of centrifugal forces.* Obviously we do not know the possible reply of Newton to objections of this kind, but we can certainly say that, making recourse to his 3rd principle of the Dynamics, Newton surely would reply that both the bucket and the universe will rotate in opposite directions with speeds respectively proportional to the inverse of their masses, or their moments of inertia[1]. We will not consider further objections of

[1] Certainly the observers on a train and on the quay of a station have no way to ascertain who is moving and who is still. But, with absolute certainty, we can apply the 3rd Principle of the Dynamics to deduce that the relative speed V , measured between the two groups of observers, is distributed among them according to the obvious relations

$$V_m = V\frac{M}{M+m}$$

and

$$V_M = V\frac{m}{M+m}$$

where m and M are respectively the mass of the train and the mass of the Earth shell that has undergone the recoil. And if, during the starting phase, the train or the mass M have undergone some plastic deformations, even from these it is possible to deduce the said speeds. We will see that the previous ratio between the interacting masses has a very important role even in gravitational phenomena, since even here the 3rd Principle of the Dynamics holds.

this kind, however we can conclude that the Principle of Action and Reaction, when it can be applied without ambiguity, allows us to state exactly what occurs in the physical world, even in cases concerning the Galileian relativity.
On the contrary, as we will see in great detail, a careful study of the Principia will show many other contradictions in Newton thoughts, and his use of some non-orthodox procedures: contradictions and procedures that are still accepted today.
It is well-known that the 3rd experimental Kepler Law states that for all the planets of the solar system the ratio (d^3/T^2) is rigorously constant.
Newton, when he considers the two-body problem, corrects this law with great opportunity and coherence by introducing the masses of the planets in it, and concludes that

$$\frac{d^3}{T^2} = \frac{GM}{4\pi^2}\left(1 + \frac{m}{M}\right)$$

where the meaning of the symbols adopted is evident.
Actually, Newton - differently from Galileo, who thinks that the bodies can only undergo the effect of the Earth gravitation but do not act on the Earth - extends the active gravitational effect to all the bodies in the universe, assuming that this effect is proportional to each single mass. Unfortunately, this powerful generalization - that has great implications, as we will see – in practice is completely voided by the procedure used in the solution of the two-body problem, when we pass from the inertial reference frame centered on the barycentre of the two masses (or the reference frame anchored to the fixed stars), to the non-inertial reference frame anchored to the central mass. Therefore, even if he recognizes that our planetary system, including the Sun, rotates around a barycentre different from the center of the Sun, Newton actually imposes an unjustified and unjustifiable heliocentrism that substitutes the controversial and fought geocentrism that, in very dark times[2], caused the death of Giordano Bruno and the abjuration of Galileo Galilei.
We will see that the removal of this unjustifiable Newtonian heliocentrism directly affects both the very small 43”- 44” per century of the shift of the perihelion of Mercury and the substantial 50” per year of the most remarkable and grand phenomenon of the lunisolar precession, and thus affects in a substantial way the entire Celestial Mechanics.

2 Some Preliminary Calculations

As a consequence of Newton’s universal theory of gravitation, two celestial bodies, at a distance d, with masses M and m (Fig. 1),

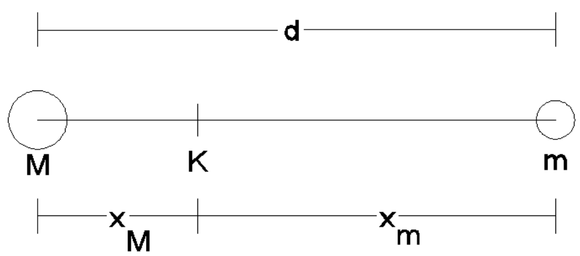

Fig. 1

in an inertial reference frame centered on the barycentre K of the two masses (or in an inertial reference frame anchored to the fixed stars), are affected by the force

[2] These times are not finished, each age has its own dark times.

$$F = G\frac{Mm}{d^2}. \tag{1}$$

Using the symbols x_M and x_m to denote the distances of the two masses from the baricentre K, we have

$$x_M = \frac{d}{M+m}m \tag{2}$$

and

$$x_M = \frac{d}{M+m}M. \tag{3}$$

In the hypothesis of circular orbits, from the two obvious relations [3]

$$\frac{MV_{MK}^2}{x_M} = G\frac{Mm}{d^2} \tag{4}$$

$$\frac{mV_{mK}^2}{x_m} = G\frac{Mm}{d^2}$$

we have the speeds V_{MK} and V_{mK} of the two masses in the frame K:

$$V_{MK} = \sqrt{\frac{Gm^2}{d(M+m)}} \tag{5}$$

and

$$V_{mK} = \sqrt{\frac{GM^2}{d(M+m)}}. \tag{6}$$

In the reference frame centered on the primary mass M we will have that the speed of the secondary mass is given by

$$V_{mM} = \sqrt{\frac{Gm^2}{d(M+m)}} + \sqrt{\frac{GM^2}{d(M+m)}} = \sqrt{\frac{GM}{d}\left(1+\frac{m}{M}\right)}. \tag{7}$$

We obtain the same conclusions by evaluating first the Newtonian force in the non-inertial reference frame centered on the primary mass. The accelerations relative to K are

$$a_{MK} = \frac{Gm}{d^2} \tag{8}$$

and

$$a_{mK} = \frac{GM}{d^2} \tag{9}$$

Therefore, in the system centered on M, we have

$$a_{mM} = \frac{GM}{d^2} + \frac{Gm}{d^2} = \frac{GM}{d^2}\left(1+\frac{m}{M}\right) \tag{10}$$

from which it follows that [4]

$$F_{mM} = \frac{GMm}{d^2}\left(1+\frac{m}{M}\right) \tag{11}$$

From (11) we have

[3] Note that, according to the 3rd Principle of the Dynamics, it is assumed that the forces acting on the two masses are identical.

[4] Analogously, an observer anchored to the secondary mass can measure the force acting on the primary mass:

$$F_{Mm} = \frac{GMm}{d^2}\left(1+\frac{M}{m}\right).$$

$$m\frac{V_{mM}^2}{d} = \frac{GMm}{d^2}\left(1 + \frac{m}{M}\right) \qquad (12)$$

and therefore the speed of the mass m in the reference frame centered on the primary mass is given by

$$V_{mM} = \sqrt{\frac{GMm}{d^2}\left(1 + \frac{m}{M}\right)},$$

i.e. we have found again relation (7). From this relation we also find the 3rd Kepler's law, revisited by Newton, that is

$$\frac{4\pi^2 d^3}{T^2} = GM\left(1 + \frac{m}{M}\right); \qquad (13)$$

It is evident that, if we consider the fall of terrestrial bodies, from (11), using a_m to denote their acceleration relative to the Earth surface and R to denote the radius of the Earth, we have

$$ma_m = \frac{GMm}{R^2}\left(1 + \frac{m}{M}\right)$$

from which it follows that

$$a_m = \frac{GM}{R^2}\left(1 + \frac{m}{M}\right) = g\left(1 + \frac{m}{M}\right). \qquad (14)$$

From (14) we immediately deduce that the law of Galilei for the fall of bodies can be obtained only when the gravitational effects exerted by these bodies are neglected; in the same way, from (13) we obtain the 3rd Kepler's law, in the form in which it was originally stated, only when the gravitational mass of the generic orbiting planet is considered equal to zero.
In a similar way, when a pendulum is considered, we find

$$T = 2\pi\sqrt{\frac{l}{g\left(1 + \frac{m}{M}\right)}}; \qquad (15)$$

so its period is also a function of the mass of the pendulum. Since in this case, considering for example a pendulum having a mass equal to 1 Kg, we find that m/M = 1.67e-25, we can realize the reason why Newton never appreciated that the conscious and consistent correction of the 3rd Kepler's law, that anyway controls the continuous fall of the planets around the Sun, automatically implies even the correction of Galilei's law [3] and the pendulum's law. In fact, Newton, contradicting completely himself, textually states (Prop. VI { Theorem VI):
The fall of all the bodies on the Earth (taking into account the unequal delay caused by the very law resistance of the air) occurs in equal times, as already observed by others; and it is possible to verify with great precision the equality of these times in pendulums. I have made experiments with pendulums made of gold, silver, lead, etc. ... .
In the following he states:
Let us imagine, in fact, that these terrestrial bodies were rosen up to the Moon's orbit, and together with the Moon, deprived of any movement, were let free so that all of them fall in the same time on the Earth; then, because of the previous reasoning, it is certain that they, together with the Moon, would cover equal spaces in equal times.
… And for the same reason, the planets that revolve around the Sun, letting them fall from equal distances from the Sun, would cover, during their fall towards the Sun, equal spaces in equal times.
(Even the planets whose masses are contained in his 3rd Kepler's law). In Prop. X { Theorem X, concerning the same subject, he states (Newton's tube) … *the bodies fall within the tube freely and without any appreciable resistance; a piece of gold and a very light feather, let free at the same time, fall with equal speed, and even if they cover an height of four, six, or else eight feet, fall contemporaneously on the bottom, as we can learn from the experience.*

We must admit that, watching with our eyes that a feather and a piece of gold fall, in a vacuum tube, with the same speed is a very impressive fact[5] .

3 Criticism of the Two-Body Problem

After having analysed the possible trajectories that a body can follow if it is attracted by a fixed center even with a law proportional to the inverse of the squared distance, Newton faces the two-body problem, i.e. the basis of the modern theory of planetary perturbations. In Prop. LVII. Theorem XXI he textually states:

If two bodies attract each other with forces of any kind and simultaneously rotate around the common center of gravity, I say that because of the effect of these forces a trajectory can be covered, around one or the other body not in motion, that is similar and equal to the trajectory that the bodies so moved cover mutually one around the other.

Fig. 2, that reproduces the original figure in the Principia, shows graphically the concept expressed by Newton.

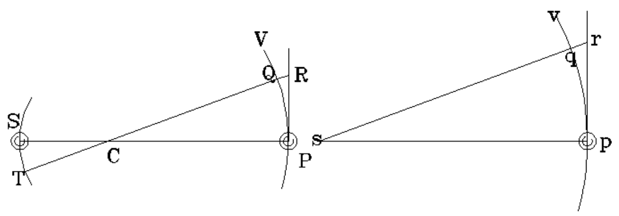

Fig. 2

On the left hand side, we see the bodies S and P that rotate around the common center C; it is evident that Newton considers the inertial reference frame anchored to the baricentre or to the fixed stars. On the right hand side, in small letters, the same figure shows the situation like it is perceived by the non-inertial observer centered in the Sun, who, according to Newton, can be considered absolutely fixed even if it is accelerated by the secondary mass. Actually, the figure on the right can be obtained from the figure on the left by tracing in the latter the straight line parallel to the line T – R and passing by S. This assumption, that, as we already said, is the basis of modern Celestial Mechanics, gives rise to the following procedure for the solution of the two-body problem.

We consider the Sun as it was absolutely fixed, by thinking to the mass of the system (M + m) as it was completely concentrated in the center of the Sun, and assuming that the secondary mass m was completely negligible [4]. This also implies that the major semi-axis of the orbit relative to the non-inertial reference frame is always fixed.

Let us refer now to Fig. 3. In this figure solid lines represent the two orbits covered by the masses M and m around the baricentre K. Furthermore, there is a frame made up by broken lines, whose intersections can be seen as a schematic representation of the fixed stars scenario. Thus it is evident that the mass m, for example, after half an orbit will be found in the point 1, while the mass M will be found on the right of its original position at a distance of about three units from the point where it was before. On the other hand, if we accept the solution by Newton, after the said time the mass m will be found in the point 2, characterized by another star constellation, while the mass M is still in

[5] If is true that for a common weight the rate m/M = 1.67e-25 is extremely short is also true that in the solar system things are totally different. In fact for the couple Sun-Mercury m/M = 1.65e-7; for the couple Sun-Jupiter m/M = 1e-3 and for couple Heart-Moon m/M =1e-2 !

its original position. Thus, if we accept the classical solution, the orbits covered in the two reference frames are completely different. In fact, while in the frame anchored to K the mass m covers the circumference with radius x_m, the same mass in the system anchored to S covers a circumference with radius x_m (1+m/M)=d. On the other hand, when, for example in geometry, one constructs the transformation formulae that relate one reference frame to the other, the guide rule is that the trajectory remains the same. We can certainly state that, with or without the presence of the fixed stars, the observer anchored to the mass M, contrary to the observer centered in K, can measure the acceleration of the mass M: this acceleration characterizes and distinguishes the two reference frames. This is well-known to the astronauts when they, during their aptitude tests, are strongly accelerated in pods rotating at high speed.

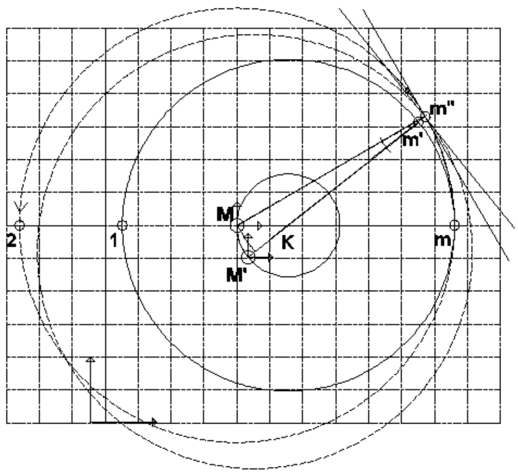

Fig. 3

Thus a problem arises: the determination of a procedure that enables the observer centered on the primary mass to reconstruct the real and unique orbit covered by the secondary mass m in the reference frame anchored to the fixed stars, and to recognize, among the other things, that he has an acceleration.
A solution that satisfies the previous requirements can be the following.
If we admit for the moment, according to Newton, that the force in the reference frame centered on the primary mass is given by the relation

$$F_{mM} = \frac{GMm}{d^2}\left(1 + \frac{m}{M}\right);$$

we deduce that, when the mass m is negligible compared with M, the reference frame centered on the primary mass is practically inertial and therefore, according to Newton, can be considered fixed. From this we deduce that this infinitesimal or fictitious mass, in the reference frame centered on the primary mass M, has a speed equal to

$$V_{mM_o} = \sqrt{\frac{GM}{d}}. \tag{16}$$

Now let us assume that the mentioned infinitesimal mass, after a prefixed time ΔT, due to the speed given by (16), starting from its initial position m, describes the arc m-m”, with radius d and center M, that can be seen in Fig. 3. Once this operation has been done, since in the given time the mass M moved to M’, it is necessary to rotate, at the same time, both the semiaxis M-m” and the small piece of the orbit covered by the arc m-m”, by the angle Mm”M’. This is needed because, contemporaneously, the abovementioned semi-axis should coincide with (or should be parallel to) the real semi-axis M’-m’ and the described arc should better approximate the orbit covered by the real mass m around the baricentre. It is evident that the rotation needed to obtain that M was very

close to M’ has the revolution direction of the orbiting mass.
In Fig. 4 this procedure is repeatedly applied in the first quadrant. This construction shows that the consecutive arcs covered by the mass m approximate the orbit covered by m around the baricentre, and that, contemporaneously, the broken line covered by the primary mass, due to the repeated rotations of the semiaxis, approximates the orbit actually covered by M around the barycentre[6]. It is evident that, choosing infinitesimal times, both the said arcs and the broken line will tend to envelope, and coincide exactly with, the two orbits around K.
Thus we can say that the non-inertial observer centered on the mass M can retrace the real orbit described by the secondary mass as long as he does the following simple operations:

- first he has to find the orbit described by an infinitesimal mass m around the mass M; in such a case he con consider himself, in every respect, like an observer fixed in the space, according to Newton; in this way he obtains, for example, an ellipse whose major semiaxis is fixed in the space;

- subsequently he imposes to the previous orbit a rotation such that its major semiaxis would be rotated, in the direction of the revolution, by an angle that we will evaluate.

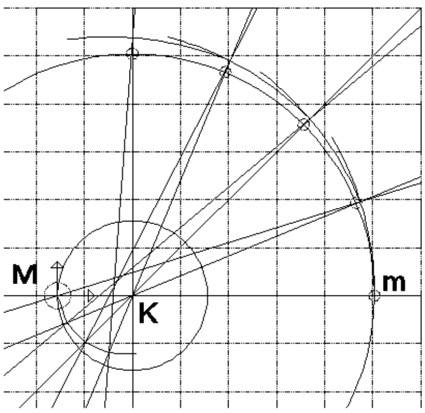

Fig. 4

In the provisional hypothesis that the point m” of Fig. 3 is very close to the point m’, like we can deduce both from Fig. 3 and Fig. 4, after a complete revolution of the mass m, the mass M has covered the whole circumference of radius x_M, so the mentioned angle, expressed in radiants, would be equal to

$$\square = \frac{2\pi x_M}{d} = 2\pi \frac{md}{(M+m)d} = 2\pi \frac{m}{(M+m)}. \tag{17}$$

Actually the angle, as we will see soon, is exactly one half of this value. To see this, let us note that since the arcs described by the masses are respectively proportional to the speed of each mass, they can represent the speed of the masses.
We have already said that the speed of the mass m, considered infinitesimal, is equal to

$$V_{mM_o} = \sqrt{\frac{GM}{d}}. \tag{18}$$

Instead, the speed of the real mass m in the reference frame anchored to the baricentre is given by

[6] In Fig. 4 we show only the rotations of the various vector radii and not the rotations of the various arcs: this has been done to clarify the figure. On the other hand, in Fig. 3 we represent with broken lines two circumferences: the lower one is the circumference characterized by the arc m”2, rotated by the angle Mm”M’.

$$V_{mK}=\sqrt{\frac{GM^2}{d(M+m)}}=\sqrt{\frac{GM}{d}}\sqrt{\frac{M}{(M+m)}}=\sqrt{\frac{GM}{d}}\sqrt{\frac{1}{\left(1+\frac{m}{M}\right)}},$$

Taking into account that the ratio (m/M) is very close to zero, this speed is also given by

$$\sqrt{\frac{GM}{d}}\sqrt{\frac{1}{\left(1+\frac{m}{M}\right)}}\approx\sqrt{\frac{GM}{d}}\sqrt{1-\frac{m}{M}}\approx\sqrt{\frac{GM}{d}}\left(1-\frac{1}{2}\frac{m}{M}\right)$$

and thus we have

$$V_{mK}\approx\sqrt{\frac{GM}{d}}-\frac{1}{2}\frac{m}{M}\sqrt{\frac{GM}{d}}. \tag{19}$$

This speed of the secondary mass m, in the baricentre system, is less than the speed given by (18), and is represented in Fig. 5 by the arc m - m', because the point m' is relative to the orbit described in the K reference frame. Instead, the speed of the real mass m in the reference frame of the fixed mass M is given by[7]

$$V_{mM}=\sqrt{\frac{GM}{d}\left(1+\frac{m}{M}\right)}\approx\sqrt{\frac{GM}{d}}\sqrt{\left(1+\frac{m}{M}\right)}\approx\sqrt{\frac{GM}{d}}\left(1+\frac{1}{2}\frac{m}{M}\right)$$

or

$$V_{mM}=\sqrt{\frac{GM}{d}}+\frac{1}{2}\frac{m}{M}\sqrt{\frac{GM}{d}} \tag{20}$$

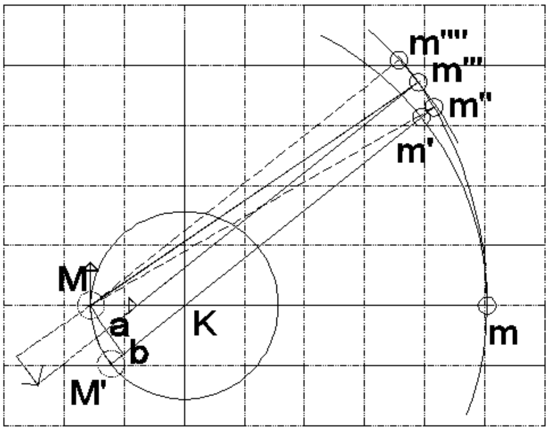

Fig. 5

and is represented by the arc m-m''''; it coincides with the mass given by the Newtonian procedure, indeed the straight line M–m'''' is parallel to the straight line M'-m'. By comparison of these relations we deduce that the speed of the mass m, considered negligible, in the inertial reference frame centered on the mass M, given by (18), is represented by the arc m – m'''. Indeed the considerations above allow us to deduce that the arc m''–m''' is equal to the arc m'''– m'''' and both represent the speed $\frac{1}{2}\frac{m}{M}\sqrt{\frac{GM}{d}}$, which implies that the arc m''-m'''', that is equal to the arc M-M',

[7] This speed is the sum of the speeds of the two masses in the K frame.

represents the speed $\frac{m}{M}\sqrt{\frac{GM}{d}}$. This implies that the rotation angle that must be applied to the orbit covered by the negligible mass to make its major semiaxis parallel[8] to the semiaxis relative to the baricentre of the masses, is given by the angle M-m’’’ a that is exactly equal to one half of the angle Mm’’M’. In fact these two angles defines the two arcs Ma and Mb whose ratio is 1/2. Therefore we conclude that, after a complete revolution of the infinitesimal mass around M, which at the same time is considered fixed, we must rotate the major semiaxis of this orbit by the angle given by

$$\alpha = \pi \frac{m}{M+m}. \tag{21}$$

This relation implies immediately that, when the secondary mass is negligible, i.e. in the case when the reference frame centered on the primary mass can be considered completely inertial, there is no need to apply any rotation to the semiaxis of the orbit determined in such a way; only in this case the perihelion of the orbit is fixed in the space.

We can obtain the same result in a more straightforward way. According to Newton, the force in the primary mass reference frame is given by

$$F = G\frac{Mm}{d^2}\left(1+\frac{m}{M}\right) \tag{22}$$

so the speed of the secondary mass m, in the primary mass frame, is equal to

$$V_{mM} = \sqrt{\frac{GM}{d}\left(1+\frac{m}{M}\right)} \approx \sqrt{\frac{GM}{d}} + \frac{1}{2}\frac{m}{M}\sqrt{\frac{GM}{d}}. \tag{23}$$

It is clear that, if the orbiting mass m is negligible compared with the central mass, the reference frame centered on the primary mass M is rigorously inertial and the secondary mass describes a fixed orbit in the space. Its speed will be obviously given by

$$V_{mM} = \sqrt{\frac{GM}{d}}.$$

When the secondary mass m is not negligible, its speed will be given by (23). It is also evident that the difference between the above-mentioned speeds

$$\Delta V = \sqrt{\frac{GM}{d}\left(1+\frac{m}{M}\right)} - \sqrt{\frac{GM}{d}} \approx \frac{1}{2}\frac{m}{M}\sqrt{\frac{GM}{d}} \tag{24}$$

is due completely to the gravitational action of the secondary mass on the primary mass, that only produces a recoil of the mass M. More precisely, we have

$$\frac{dV_{mM}}{dm} = \frac{1}{2}\frac{m}{M}\sqrt{\frac{GM}{d\left(1+\frac{m}{M}\right)}} \tag{25}$$

which implies

$$\Delta V = \frac{1}{2}\frac{m}{M}\sqrt{\frac{GM}{d\left(1+\frac{m}{M}\right)}}. \tag{26}$$

The length of the arc covered by the real mass m with the speed given by (26) on the circumference of radius d during a whole period of revolution T is given by

$$s = \frac{1}{2}\frac{m}{M+m}\sqrt{\frac{GM}{d\left(1+\frac{m}{M}\right)}}\sqrt{\frac{4\pi^2 d^3}{GM\left(1+\frac{m}{M}\right)}} = \pi\frac{m}{M+m}d \tag{27}$$

[8] It would be coincident if m’’’ coincided with m’’.

and thus

$$\square = \square \frac{m}{M+m}.$$

All the above is schematically shown in Fig. 6.

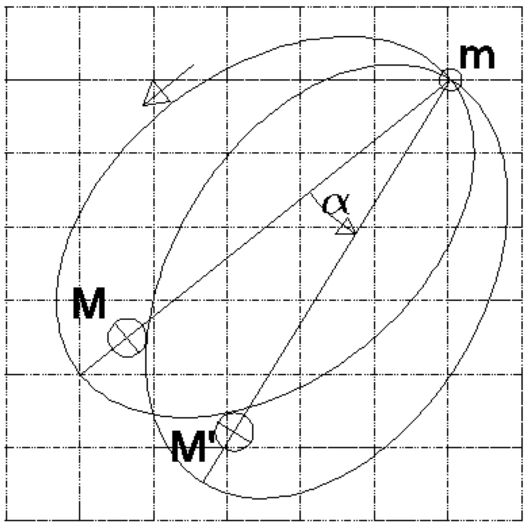

Fig. 6

When the secondary mass m has covered a complete orbit, the primary mass M assumes the position M', with the major semiaxis rotated by the angle □.

It is easy to realize that, operating in such a way in the case of circular orbits, the secondary mass, in the reference frame centered on M, will cover exactly the orbit that the real mass m covers around K with the speed given by (18). At the same time the circular orbit itself will be characterized by a slide speed, in the same direction of the revolution, given by (26). In conclusion the secondary mass will have a total speed equal to which is equal to the speed of the real mass in the Newtonian reference frame. In fact we have

$$V_{mM} = \sqrt{\frac{GM}{d}\left(1+\frac{m}{M}\right)} \square \sqrt{\frac{GM}{d}} + \frac{1}{2}\frac{m}{M}\sqrt{\frac{GM}{d}}. \tag{28}$$

We can conclude that the speed of the secondary mass in the reference frame centered on the fixed primary mass of Newton can be reinterpreted by considering two parts:

- the so-called Galileian part represented by the formula $\sqrt{GM/d}$, which does not depend on the orbiting mass and generates an orbit fixed in the space;
- and the speed part given by the relation $(m/(2M))\sqrt{GM/d}$, which causes the rotation of the orbit, which is proportional to the secondary mass.

4 Formula for the Inertial □ Non-Inertial Reference Frame Transformation

When Newton, still very young, had the genial intuition that the acceleration of the Moon and the acceleration of the bodies on the Earth's surface were connected by the relation of the inverse square of distance, he needed to check his idea with an experimental test. Unfortunately the data available to him were not sufficiently precise and he was forced to leave his attempt. Much time later, having obtained more reliable astronomic data, he made another attempt. Contrary to his original expectations, he found that there is not a perfect equivalence between his hypothesis and the reality, since, for a real equality the law should be of the type

$$F = \frac{Const.}{d^{2.016731516}} \tag{29}$$

instead of an exponent of d exactly equal to 2, as he assumed. This fact was a problem also for Cotes, author of the original introduction to the Principia. Indeed, Cotes, to avoid possible criticism, states that it is true that in the case of the Moon the law of the inverse square is not completely satisfied, but it is also true that the deviation can be justified by considering the perturbations caused mainly by the Sun. This deviation is given by Newton in an incomplete way, justifying only a part of it. Just in this context, continuing his research in this direction, with one of the most valuable pages of his masterpiece, and using also his so-called fluxion calculus (today's differential calculus), he shows the following (Prop. XLV { Problem XXXI).
The exponent of d in the equation (29), for an elliptical orbit with small eccentricity, is a function of the forward shift or backdating of the periastron or, as it was usual to say at the time, of the apses of the orbit, according to the relation

$$3 - \left(\frac{360°}{360° + \alpha°}\right)^2 \equiv 3 - \left(\frac{2\pi}{2\pi + \alpha^{rad}}\right)^2 . \tag{30}$$

Obviously, if the forward shift or backdating is zero, the exponent of d is exactly equal to 2. Corollary 1 of Prop. XLV, always concerning the Moon, states textually that *The centripetal force, thus, decreases according to a proportion slightly greater than the square, but 59:75 times closer to the square than to the cube.* Even here we see the care expressed by Cotes in the preface of the Principia.
But we have to say that even the perihelia, or the apses of the planets, are not fixed but have a considerable shift in the same direction of the revolution of the planets around the Sun. In particular, the lunar perigee has a more remarkable shift than the perihelia of all planets. This fact is already a question point. Let us observe that, while the ratio between the mass of Jupiter, the biggest planet of our system, and the Sun is equal to 1/1040, the ratio between the mass of the Moon and the mass of the Earth is much higher, being equal to about 1/81; this let us forecast, on the basis of the previous arguments, a strong recoil of the Earth because of the gravitational mass of the Moon. Having said that, taking into account that the apses of the Moon have a forward shift equal to 3° 3' in a complete revolution of about 27 days, then, in such a case, we find that the exponent of d coincides with the exponent given by (29). Newton, with the well-known two-body procedure, which disregards the movement of the primary mass, finds for all the planets and therefore even for the moon, an orbit absolutely fixed in the space. Consequently he must explain, by means of other causes, a discrepancy of the exponent of the mean radius whose value is 0.01646, just equal to all the observed 3° 3'. He attributes this value to the influence of the Sun on the lunar orbit, but, as we have already said, he is only able to justify a part of the said discrepancy (Prop. LXVI { Theorem XXVI etc.). Many years after, the French astronomer and mathematician Clairaut (1747) stated officially that the lunar motions are incompatible with the Newtonian theory of gravitation [7], obtaining even the agreement of Euler, and suggested the existence of an additional force besides the Newtonian force, inversely proportional to the cube of the distance. In a second instance, even more sensationally, he denied himself. When he took part in the Competition announced by the Academy of St. Petersburgh he showed that what he had previously stated was not true, maybe only with the purpose to win. And still today, the Moon has some completely unexplainable motions [4, p. 222].
In view of the previous considerations, the Earth, because of the lunar mass, must move back[9]. This backward movement can be interpreted by the terrestrial observer, as already said, like a forward shift of the lunar perigee. Therefore we have

[9] Note that the mass of the Moon has been determined just by taking into account that the Earth-Moon system orbits around the common baricentre.

$$3\cdot\left(\frac{2\pi}{2\pi+\theta^{rad}}\right)^2=\left(\frac{1}{1+\frac{1}{2}\frac{m}{M+m}}\right)^2\approx \qquad (31)$$

$$\approx 3\cdot\left(1-\frac{1}{2}\frac{m}{M+m}\right)^2\approx 3\cdot\left(1-\frac{m}{M+m}\right)=2+\frac{m}{M+m}\approx 2+\frac{m}{M}$$

and so we obtain

$$F_M=G\frac{Mm}{d^{2+\frac{m}{M+m}}}\approx G\frac{Mm}{d^{2+\frac{m}{M}}} \qquad (32)$$

or

$$F_M\cong G\frac{Mm}{d^{2.012161826}} \qquad (33)$$

with a residual, equal to 0.045697, that must be justified by other causes. Formula (32) can be seen like the transformation of Newton's law when the observer anchored to the inertial reference frame of the mass baricentre is substituted by the observer centered on the primary mass.

5 The Shift of Mercury Perihelion

In this case (32) becomes

$$F_S=G\frac{Mm}{d^{2.000\,000\,165}} \qquad (34)$$

which in practice is the same as the correction of Newton's law previously proposed by the astronomer Hall. He evidently starts from the assumption that the shift of Mercury perihelion is equal to 43'' per century and, using Newton's relation, concludes that the exponent of d is equal to 2.000 000 16 for all the planets.

Actually, according to (21), we deduce that the apparent forward shift of the perihelion of Mercury due to the real backward shift of the Sun, along a revolution period of 88 days, is equal to

$$\Delta^\circ=180^\circ\frac{m}{M+m}=180^\circ\cdot 0.000\,000\,165=0.000\,029\,728.$$

In connection with the shift of the perihelion of Mercury we must do some considerations about some inconsistencies contained in the calculations of Le Verrier and Newcomb. The meticulous French astronomer applies literally the theory of arbitrary constants by Laplace, father of the modern theory of perturbations. This theory, among other things, has also the purpose of determining the unknown masses of some planets starting from the distortions that these masses induce on the Keplerian orbit (**fixed in the space**) of another planet. Since Mercury has no satellites, he was forced to deduce the approximate amount of its mass by means of a number of indirect checks. In its work [4] he textually states:

Dans plusieurs recherches, j'ai reduit cette masse a 1/3000000 (of the solar mass), en consideration des pertubations qu'elle a fait eprouver a la comete d'Encke, dans son passage au perihelie, en 1838. Mais, suivant M. Encke, le masse de Mercure serait encore plus faible, et egale a 1/5000000 de la masse du Soleil. ***Nous conclurons donc seulement que cette masse est fort petite, et qu'elle ne peut avoir aucune influence sensible sur le calcul du grand axe de l'orbite***[10].

At this point we need a clarification. The data that an astronomer can use to determine the characteristics of an orbit are, among the others, the period of revolution and the mass of a planet.

[10] We have already seen (see Fig. 3, too) that the two-body procedure of Newton requires that the secondary mass describes around the Sun, considered fixed, an orbit having a mean radius greater than the radius of the orbit covered around the common baricentre, and exactly equal to the latter amplified by the coefficient (1 + m/M).

Knowing these data it is possible to calculate the semi-axis a of the orbit of a planet through the well-known relation

$$T = \sqrt{\frac{4\pi^2 a^3}{GM\left(1+\dfrac{m}{M}\right)}}, \tag{35}$$

The astronomers write down (35) in the following form

$$n = a^{-3/2}(1+m)^{1/2}\sqrt{k} \tag{36}$$

assuming that n is the arc, expressed in sexagesimal seconds, that the planet covers in 24 hours, that the mass of the Sun is equal to 1, and that m is equal to the ratio of the mass of the planet and the mass of the Sun. Furthermore they assume that a is the major semi-axis of the planet orbit, that the Earth-Sun distance is equal to 1, and that $\sqrt{k} = 3548.18760696510$ of sexagesimal arc [6, p. 198].

When Le Verrier states that the mass of Mercury is negligible compared with the mass of the Sun, he just wants to state that no appreciable errors are done. This undemonstrated certainty makes explicit reference to the following statement by Laplace [5].

We have determined, [5, Chapter V], the arbitrary constant quantities, so that the mean motion and the equation of the center may not be changed by the mutual action of the planets. Now we have, in the elliptical hypothesis,

$$\frac{1+m}{a^3} = n^2,$$

the mass of the Sun being put equal to unity. Hence we obtain

$$a = n^{-2/3}(1+m)^{1/3} \cong n^{-2/3}\left(1+\frac{1}{3}m\right) \tag{37}$$

for the semi-transverse axis, which must be used in the elliptical part of the radius vector. If we suppose, in conformity to the principles assumed in (4078-4079, etc), that

$$a \cong n^{-2/3}$$

(thus the gravitational masses of the planets are neglected) *we must increase a, a', etc., in the calculation of the elliptical part of the radius vector by the quantities (1/3)m etc.;* ***but this augmentation is only sensible in the orbits of Jupiter and Saturn.***

Thus, according to Laplace, in the case of Mercury, it is possible to neglect its mass. **Let us see if this is true. We will demonstrate that this gratuitous assumption is completely false and noxious**. From (36) we have

$$\frac{dn}{dm} \cong \frac{1}{2}a^{-3/2}\left(1-\frac{1}{2}m\right)\sqrt{k}$$

from which it follows

$$\Delta n = \frac{1}{2}ma^{-3/2}\left(1-\frac{1}{2}m\right)\sqrt{k} \cong \frac{1}{2}ma^{-3/2}\sqrt{k} \tag{38}$$

Assuming the following values [6] [8]:

$$a = \frac{57.91e11}{146.467e11} = 0.3874433$$

$$\mathrm{m} = 0.000000165$$

(38) gives the following value

$$\Delta n = 0.0012138''.$$

This is the angle subtended by the arc of the orbit, expressed in sexagesimal seconds, for a mass variation equal to m, in 24 hours. In a century we have

$$\delta = 0.0012138 \times 36524 = 44.33''.$$

To be clearer, given Newton's results, let us suppose that Mercury covers a circular orbit around the Sun (Fig. 7). Let us admit, for the moment, that Mercury has a completely negligible mass. As a consequence it runs along its orbit with the following speed:

$$V_1 = \sqrt{\frac{GM}{d}} = \sqrt{\frac{6.67e-8 \cdot 1989e30}{57.91e11}} = 4786339.784 [cm/\sec] \qquad (39)$$

On the other hand, if we take into account even its mass, we will have:

$$V_2 = \sqrt{\frac{GM}{d}} \sqrt{\left(1 + \frac{m}{M}\right)} = 4786339.784 \cdot \sqrt{1.000000165}\ [cm/\sec] \qquad (40)$$

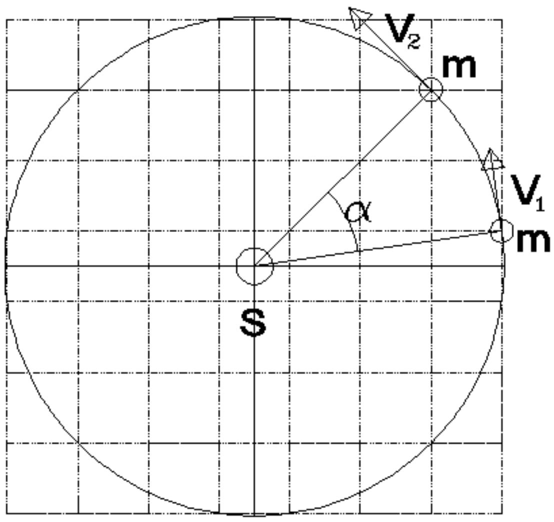

Fig. 7

Thus there is a speed difference equal to

$$\Delta V = 0.39442\ [cm/\sec]$$

These two hypothetical bodies that run on the same orbit with distinct speeds, after a period of 88 days, equal to the period of revolution of Mercury, are separated by a distance given by the arc

$$s = 0.39442 \cdot 88 \cdot 24 \cdot 3600 = 2992254.144\ [\text{cm}]$$

After a century they will be separated by the arc

$$s = \frac{2998854.144}{88} 36524 = 1244660781\ [\text{cm}]$$

which is equivalent to an angle centered in the Sun equal to

$$\alpha^{rad} = \frac{1244660781}{57.91e11} = 0.00021493$$

or else equal to

$$\alpha'' = \frac{0.00021493 \cdot 180}{\pi} = 44.33''.$$

This is equivalent to say that, in view of the theory of Newton, to neglect or to not know the secondary mass implies, in the case of Mercury, an error practically equal to the supposed forward shift[11].

We already stated at the beginning of this argument that Le Verrier assumes that Mercury has a mass (expressed in grams) comprised in the interval

$$\Delta m = [3.978e26, \quad 6.63e26].$$

---

[11] Actually it is a residual shift of the perihelion of Mercury that cannot be explained by Newton's theory.

If we make all the calculations with these values, we have an uncertainty in the determination of the planet equal to 37" in a century, practically equal to the 38" that he, with great resonance, communicated to the scientific community of his age, obviously thinking to the influence of the unexisting planet Vulcan.
The work by Newcomb is only slightly different; he, at the end of the calculations, deduces for Mercury a mass comprised between the limits

$$\frac{1 \pm 0.35}{7900000}$$

i.e.

$$\Delta m = \left[3.4e26, \quad 1.64e26\right]$$

that implies an uncertainty per century equal to 24". To completely appreciate what we have just said, we must first state the following. Actually, even with the two-body procedure of Newton, the secondary mass m is taken into account. In fact, as we already said, from Fig. 3 we deduce that the fictitious orbit that the mass m covers around M, assumed fixed, has a radius consequently given by

$$d = x_m \left(1 + \frac{m}{M}\right) \tag{41}$$

where x_m is a radius that can be called the real radius of the orbit covered by m, since it is covered around the baricentre K. Thus, when we neglect the mass of the planet, the length of the semi-axis of the orbit does not change. On the other hand, nothing prevents us to think that the radius of the orbit of two masses is fixed, if one is equal tom and the other is zero. In such a case the increase of the radius neglected by Le Verrier becomes an arc increase orthogonal to it [12]. The latter becomes a localization error of the planet on its orbit, if we want to stay still in the context of the two-body problem like it was originally set by Newton, that coincides with the missing shift of Mercury's perihelion. Therefore, if the astronomical calculations are done with these approximations, we are forced to accept the consequent and strong implications with the greatest caution. On the other hand, if we admit the physical existence of the recoil, we have that the above-mentioned radius increase or speed increase become completely a real backward shift of the central mass or an apparent forward shift of the perihelion of the orbit considered, in case we still would continue to think of a central mass fixed in the space.

6 Integration of the Equation and the Orbital Spin

In polar coordinates we can write

$$m\left(\ddot{r} - \frac{h^2}{r^3}\right) = -\frac{GMm}{r^{2+\frac{m}{M}}}.$$

Setting, as usual, u = 1/r, we have

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm}{h^2} u^{\frac{m}{M}} = \frac{GM}{h^2} u^{\alpha} \tag{42}$$

In case of an ellipse with small eccentricity, we introduce a parameter u_o (taking into account that u = 1/r) very close to the inverse of the mean radius of the orbit. We can write:

$$u^{\alpha} = \left(u - u_o + u_o\right)^{\alpha} = u_o^{\alpha}\left[1 - \left(1 - \frac{u}{u_o}\right)\right]^{\alpha}. \tag{43}$$

Since u_o is very close to u, the term (1-u/ u_o) is very small and almost equal to zero. Then, with a good approximation, we have

[12] Laplace (see note no. 2251) states: If this quantity were an arc of the planet's orbit, perpendicular to the radius vector, it would subtend only an angle of 3. 6", when viewed from Sun (in the case of Uranus).

$$u_o^{\alpha}\left(1-\left(1-\frac{u}{u_o}\right)\right)^{\alpha} \approx u_o^{\alpha}\left(1-\alpha\left(1-\frac{u}{u_o}\right)\right)=u_o^{\alpha}-\alpha u_o^{\alpha}+\alpha u_o^{\alpha-1}u \tag{44}$$

from which it follows[13]

$$\frac{d^2u}{d\theta^2}+u\approx\frac{GM}{h^2}(u_o^{\alpha}-\alpha u_o^{\alpha}+\alpha u_o^{\alpha-1}u). \tag{45}$$

Setting

$$a=\frac{Gm}{h^2}u_o^{\alpha},\quad b=-\frac{GM}{h^2}\alpha u_o^{\alpha} \tag{46}$$

$$c=\frac{GM}{h^2}\alpha u_o^{\alpha-1}=\frac{GM}{h^2u_o}\frac{m}{M}u_o^{m/M}, \tag{47}$$

$$d=a+b \tag{48}$$

we finally have, for an elliptical orbit with small eccentricity, as mentioned above,

$$\frac{d^2u}{d\theta^2}+u\approx d+c\,u \tag{49}$$

hose solution is the following:

$$u=\frac{d}{c-1}C_1\sin\left(\sqrt{1-c}\,\theta\right)+C_2\cos(\sqrt{1-c}\,\theta)=\frac{d}{c-1}+C\cos\left(\sqrt{1-c}\,\theta-\theta_0\right) \tag{50}$$

or, in a simpler form:

$$u=\frac{d}{c-1}+C\cos\left(\sqrt{1-c}\,\theta\right)=A+C\cos\left(\sqrt{1-c}\,\theta\right). \tag{51}$$

Using polar coordinates we find

$$r=\frac{p}{1+\varepsilon\cos\left(\sqrt{1-c}\,\theta\right)} \tag{52}$$

Differentiating and equating to zero, we deduce that after a complete revolution

$$\left(\sqrt{1-c}\,\theta\right)=2\pi\,. \tag{53}$$

Since we have

$$c=\frac{GM}{h^2u_o}\frac{m}{M}u_o^{m/M}=\left(\frac{GM}{(r_ov_o)^2}r_o\right)\frac{m}{M}u_o^{m/M}=(1)\frac{m}{M}u_o^{m/M}=\frac{m}{M}u_o^{m/M} \tag{54}$$

and, from (53),

$$\theta=\frac{2\pi}{\sqrt{1-c}}=\frac{2\pi}{\sqrt{1-\frac{m}{M}u_o^{m/M}}}\approx 2\pi\left(1+\frac{1}{2}\frac{m}{M}u_o^{m/M}\right) \tag{55}$$

and since we also have (in the case of Mercury or the other planets)

$$u_o^{m/M}=u_o^{0.000000165}\approx 1$$

we find from (55) that, after a complete revolution, the rotation angle of the axis of the orbit is

$$\theta=2\pi\left(1+\frac{1}{2}\frac{m}{M}\right)=2\pi+\pi\frac{m}{M}$$

and thus it is increased of the quantity

$$\Delta\theta=\pi\frac{m}{M} \tag{56}$$

Then (52) can also be written as

[13] Osserviamo lo sviluppo in serie del binomio di Newton troncato al primo termine.

$$r = \frac{p}{1 + \varepsilon \cos\left(\theta \sqrt{1 - \frac{m}{M}}\right)}. \tag{57}$$

Thus the orbit is characterized by a spin given by

$$\frac{d\omega}{dt} = \dot{\omega} = \frac{\pi}{T}\frac{m}{M+m} = \frac{\pi}{\sqrt{\frac{4\pi^2 d^3}{GM\left(1+\frac{m}{M}\right)}}}\frac{m}{M+m} = \frac{1}{2}\frac{m}{M}\sqrt{\frac{GM}{d^3\left(1+\frac{m}{M}\right)}}. \tag{58}$$

In the case of the Sun-Mercury pair, in a century, we have a rotation of the major semi-axis of the orbit towards the point γ, that, in sexagesimal seconds, is approximately equal to

$$\frac{\omega^{rad}}{\sec} = \frac{0.3285e27}{2 \cdot 1989e30}\sqrt{\frac{6.67e-8 \cdot 1989e30}{(57.11e11)^3 \cdot \left(1+\frac{0.3285e27}{1989e30}\right)}} \approx 6.96e-14$$

or

$$\frac{\omega''}{100} = 6.9692e-14 \cdot \frac{180}{\pi} \cdot 3600 \cdot 36524 \cdot 86400 \approx 45''$$

On the other hand, in the case of the Sun-Jupiter pair, **just in 1 year**, we have

$$\frac{\omega^{rad}}{\sec} = \frac{1897.1e27}{2 \cdot 1989e30}\sqrt{\frac{6.67e-8 \cdot 1989e30}{(778.34e11)^3 \cdot \left(1+\frac{1897.1e27}{1989e30}\right)}} \approx 7.99e-12$$

$$\omega''/1 = 7.9955e-12 \cdot \frac{180}{\pi} \cdot 3600 \cdot 365.24 \cdot 86400 \approx 52''.$$

Finally, let us observe that, if we find the force law from (57), we have:

$$F_M = -\frac{mh^2}{p}u^2\left(1 - \frac{m}{M} + \frac{m}{M}pu\right) \approx -\frac{mh^2}{p}u^2\left(1 + \frac{m}{M}pu\right).$$

This relation becomes

$$F_M = -\frac{mh^2}{p}u^2\left(1 + \frac{m}{M}pu\right) = -\frac{GMm}{d^2} - \frac{m}{M}p\frac{GMm}{d^3} + .. + \tag{59}$$

and therefore the observer centered in the primary mass sees the action of an additional force inversely proportional to the cube of the distance on the secondary mass (isn't it that one of Clairaut ?). If we assume that the *constant p* and the *variable d* are almost the same, we obtain again Newton's formula:

$$F_M = -\frac{GMm}{d^2} - \frac{m}{M}p\frac{GMm}{d^3} = -\frac{GMm}{d^2}\left(1 + \frac{m}{M}\right).$$

7 On the Theory of Planetary Perturbations

According to Newton's theory, we deduce that, if the Moon would fall on the Earth, even the Earth, affected by the lunar gravity, would move towards the Moon. Thus an earthly observer would have an acceleration and therefore a measurable movement compared with the fixed stars, that certainly are not involved in the said phenomenon. The same thing continuously occurs for the Sun that, due to the different planetary masses, undergoes continuous accelerations that force it to move in the same direction of the planets, around the common center of gravity. Thus an observer centered on a given planet should see the Sun that moves backward with its own motion on the relevant ecliptic, thus going towards the γ point with a speed that is a function of the secondary mass considered. The following table shows the various backward movements, expressed in sexagesimal seconds, calculated by means of one of the previous formulae, that each planet causes to the Sun (whose mass is assumed equal to 1989e30 [g] [8]), or the apparent forward shift of the perihelion of the planet, in a century.

Planet	Distance	Mass	Revolution	Backward shift
	cm [□*e*11]	g[□*e*27]	days or years	per century
Mercury	57.91	0.3285	88	44''
Venus	108.21	4.8714	224.7	258
Earth	149.467	5.976	365.24	195
Mars	227.94	0.645	687	11
Jupiter	778.34	1897.1	11.86	5206
Saturn	1427	567.7	29.46	627
Uranus	2869.6	86.7	84.01	34
Neptune	4496.6	105.2	164.8	21
Pluto	5947	0.00989	247.7	0.0013

It is immediately evident that Jupiter causes approximately, in a year, a 52'' backward shift, and the Earth a 2'' backward shift. The difference is equal to 50", and is, strangely, exactly same that should be due to the lunisolar precession. Before we describe a deeper study of this subject, let us give an insight to the theory of planetary perturbations. In the following table we show, for each planet, the perihelion shift measured starting from the □point, temporarily considered fixed in the space, along a century [6].

Planet	Perihelion forward shift in seconds of arc per century
Mercury	5600.821''
Venus	5064.415''
Earth	6190.672''
Mars	6627.197''
Jupiter	5799.664''
Saturn	7053.272''
Uranus	5344.819''
Neptune	5129.875''

The reference frame adopted to study the motion of the planets of the solar system has its origin in he center of the Sun, **obviously considered fixed in the space**, and an axis directed in the □point, that is one of the intersections of the ecliptic and the equatorial plane. The major semi-axis of the ellipses covered by the various planets is not fixed compared with the □point, but it has a forward shift in the direction of a backward revolution, covering the amount shown in the table. So let us see the way in which these forward shifts are justified in the Classical Celestial Mechanics.

Stating in advance that in a year there is a relative movement, between the Sun and the □point, equal to approximately 50'' per year, and that this movement is explained by Newton as if it was a consequence, *given the assumed fixed position of the Sun*, of the whole phenomenon of the lunisolar precession, we deduce that the theory of planetary perturbations must explain, for every planet, the residual forward shift shown in the following table.

Planet	Residual forward shift in seconds of arc per century
Mercury	580''
Venus	45
Earth	1170
Mars	1607
Jupiter	779
Saturn	2033
Uranus	324
Neptune	109

It is well-known that the Newtonian theory is able to explain only 537"of the whole shift of the perihelion of Mercury. The orbit of Venus is almost circular, and therefore it is difficult to appreciate its forward shift. Perhaps for Mars a tenth of seconds cannot be explained. On the other hand if we accept the present version of the backward shift, the following residuals should still be explained:

Planet	Exponent	Residual forward shift in seconds of arc per century
Mercury	2,000.000.165	536''
Venus	2,000.002.449	-213[14]
Earth	2,000.003.004	975
Mars	2,000.000.324	1596
Jupiter	2,000.953.795	779
Saturn	2,000.285.149	1406
Uranus	2,000.043.589	290
Neptune	2,000.052.790	21

Although such a work can be done in a distinct article, we can already forecast a model that could easily give quite correct results. Actually we can admit, in a first instance, that the Sun is forced to rotate around the Sun-Jupiter baricentre with a given speed. Therefore we can assume, as a first scheme for calculations, that the system is formed by two masses, of which the primary is forced to rotate around the said center. However we must observe that the residuals that should be explained are, in both cases, approximately the same. But we must still re-examine the whole question of the precession, which we will briefy consider in the following section.

8 On the Problem of the Lunisolar Precession

There is no doubt that Newton cannot attribute all the shift (experimentally observed by the astronomers and measured by the above-mentioned 50'' of arc per year) between the Sun and the □ point to the one and only phenomenon of the precession. This is due to the fact that the Sun must move towards the □point, even by a small distance, because of the gravitational force exerted by the planets. On the other hand, Newton himself, in the third book of the Principia, textually admits (Prop. XII. Theorem XII) that

The Sun has a continuous motion, but never moves at a great distance from the common center of gravity of all the bodies.

[14] We have already said that the orbit of Venus is almost circular. In such a case, the apparent forward shift due to the real backward shift of the Sun, according to what we have already said in Section 3, is masked by the intrinsic speed of the planet, since it is difficult to highlight the slide speed of the orbit itself.

And, if this is true, the Sun must necessarily move, on the ecliptic traced by itself, in a backward direction, towards the □point by a distance that still has to be determined. On the other hand, in the hypothesis that the demonstration of Newton on the cause of the precession was completely wrong, the fact that Jupiter would force the Sun to move towards the □point exactly by (52'' – 2'') = 50" (which is the result of a calculation due to the independent determination of the masses of the Sun, Jupiter, and the Earth) should and would perhaps be the greatest success of Newton's theory.
But should the explanation of Newton about the precession be completely excluded ?
A first fact is certain.
The whole 50'' per year cannot be attributed completely to the equatorial bulge of the Earth. This item is sure and undisputable. Then in what proportions must the above-mentioned shift be distributed between the real motion of the Sun and the said effect?
This is the problem.
On the other hand it is clear that, if the precession was completely due to the motion of the Sun, the formula

$$F = \frac{GMm}{d^{2+\frac{m}{M}}} \tag{60}$$

would be absolutely exact. If, instead, the precession should be attributed completely to the equatorial bulge, the exact formula would be:

$$F = \frac{GMm}{d^2} \tag{61}$$

So it seems clear that in an intermediate case the gravitation law should be of the type

$$F = \frac{GMm}{d^{2+\square\frac{m}{M}}}$$

with a coefficient □□ [0,1] that can be determined even by repeated attempts. Another important point that should be taken into account is that our criticism of the two-body problem assumes, among the other things, that gravitational actions are instantaneous and that there is perfect equality between gravitational mass and inertial mass.
So it seems reasonable to enumerate the objections that can be moved against the interpretation of the precession done by Newton.

- The first is the already said objection concerning the proper motion of the Sun on the ecliptic, caused by the planets.
- The second is that the precession is variable and its time law, given by Newcomb's relation (1900) [15], is found to be [9]

$$50.2564 + 0.000222\ (\text{year} - 1900) \tag{63}$$

Thus, since 1900 till today, it is increased by 0,023". This increase is completely incompatible with the invariability of the masses of the Moon and the Earth (Newton (Prop. XXXIX. Problem XX) attributes □ 41" to the action of the Moon and □ 9" to the action of the Sun on the terrestrial bulge [16]). Instead, it is compatible with the evident variable composition of the different backward shifts that affect the Sun because of every planet. Actually, this composition is a function of the relative positions of the single planets (conjunctions, oppositions, quadratures, etc.). In fact, this study, never attempted, should enlighten the phenomenon, clarify the connections between the said

[15] There are certainly more precise relations, but the content is the same.

[16] Thus, if we assume that the force exerted by the Sun on the equatorial bulge is equal to 1, the analogous force by the Moon is 4 times stronger.

variabilities, and contemporaneously give a way to determine the speed of the gravitational perturbation. This possibility follows because a foreseeable variation of the previous shift due to a particular configuration of the planets could show its effects after some time. The physicists W. Cruttendon and V. Dayes [9] think that the explanation of Newton is completely inconsistent and that the said 50” per year are totally due to the proper motion of the Sun because it could be one of the component stars of a binary system. And the Sun and Jupiter do they form a binary system ?

- Another subtle consideration has been suggested by K. Homann [10]. He completely denies any validity of Newton's explanation, and attributes the said 50” to the orbital motion of the Sun around a star of the Sirius group.
- Actually, Newton's explanation bases on a similitude between the motion of the lunar nodes and the effect of lunisolar actions on the terrestrial bulge. We immediately note that this hypothesis is strongly forced: contrary to the equatorial bulge, the Moon is not strictly anchored to the Earth. Its orbit could be orthogonal to the ecliptic plane without affecting the terrestrial globe. On the contrary, if the Earth and the Moon were connected in a rigid structure, then only in that case the action of the Sun on the Moon would actually cause the wellknown gyroscopic effect. The Moon covers an orbit slightly inclined compared with the ecliptic plane. The straight line characterized by the intersection of the lunar orbit projected on the celestial sphere and the ecliptic plane determines the said nodes N and N’. These nodes advance clockwise (just as would do the ▯point) for the different actions that the Sun exerts on the Moon when it lies above or below the ecliptic plane.

The subject in the final item deserves a deepest consideration; so, to see the relative deductions in a better way, let us consider Fig. 8:

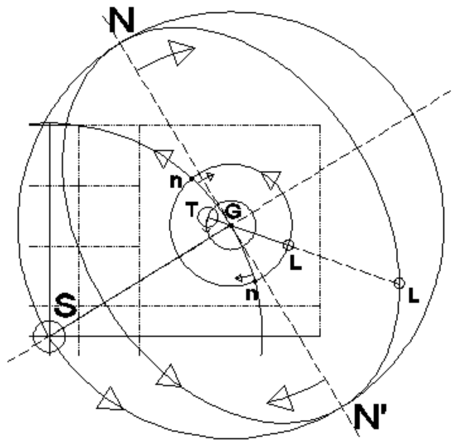

Fig. 8

where we show the orbit of the baricentre G around the Sun (the arc nGn) and the orbit of the Moon and the Earth around the common barycentre G. All these orbits are covered clockwise. The intersection nodes of the orbit of the Moon and the orbit of the Earth around the Sun, or better, of the baricentre G around the Sun, are marked by the letter *n*. In the same figure we have also shown the ecliptic (circumference of radius GS), i.e. the projection on the celestial sphere of the apparent trajectory that the Sun covers around G, and the orbit of the Moon (ellipse characterized by the arc N’LN; we have an ellipse because, although the lunar orbit is practically circular, this circumference is inclined by 5° 9’ compared with the circumference that represents the ecliptic) also projected on the celestial sphere, like it can be seen by an observer centered in the baricentre G. (Here it is clearly evident that we could make our reasoning on the nodes *nn*, instead of the nodes NN’). The lunar nodes N and N’, taken into account that the lunar orbit is assumed to be fixed - as required by the two - body problem in Newton's formulation -, move clockwise, as can be seen in the figure, and go towards the Earth that, on the contrary, is forced to move counterclockwise by the mass of the Moon around the common barycentre G. Even in this case it is quite evident that we cannot consider the Earth as it was absolutely fixed and therefore attribute the whole motion of the nodes to the nodes themselves, like we did for the motion of the □point. Instead we have seen that the central body can be considered fixed provided we attribute a clockwise rotation equal to $□ = 180° m / M$ to the axis of the orbit, and, just as in the case of the Moon, given its exceptional mass compared with the mass of the Earth, the said forward shift is at a maximum and equal to 2.18685° in 27.32166 days. On the other hand, the mass of the Moon has been determined just by means of an astronomical detection of the small ellipse that the Earth covers around the common center of gravity [11, p. 335] and thus it is impossible to deny that a proper motion of the Earth towards the said nodes exists. Instead, according to Newton, the whole forward shift of the nodes -a clockwise motion that covers a whole arc of 360° in 18.6 years and thus, in a whole period of revolution of the Moon, is equal to 1.4478°, should be attributed to the action of the Sun that, in the attempt to make coincident the lunar orbit and the ecliptic plane, causes the said clockwise motion of the nodes. Newton imagines a set of moons that form a ring [17] and thus extends the motion of the nodes to such a ring. Subsequently he thinks to a real solidification of these moons and thus imagines that this solidified ring takes the place of the equatorial bulge of the Earth. Finally Newton thinks that this ring is firmly anchored to the spherical earthly globe to which it transmits, without any reduction, the motion of the above-mentioned ring. Assigning to the Moon a force 4 times greater than the force of the Sun (assumed equal to 1), he establishes that the precession phenomenon has to be attributed to the Moon (□ 40”) and to the Sun (□ 10”). Furthermore, the Moon should exhibit a forward shift of the perigee, completely independent from the clockwise motion of the nodes, that covers 360° in 8.85 years and corresponds to a forward shift of the periastron equal to 3.0429°, always in a period of revolution equal to 27.32166 days. This forward shift, in this case, would be due only and exclusively to the other action of the Sun on the Moon that, in such a case, sometimes reduces and sometimes increases the action of the terrestrial gravitation. All of this by thinking that the Earth is absolutely fixed. On the other hand, it is well-known that Newton, even in the case of the precession, adjusts with suitable *fudge factors* [12, p. 22 and following] the parameters of the problem until there is coincidence between the theoretical solution and the experimental data. Westfall [13] has shown that Newton, even in this very special circumstance, manipulates many times the inclination of the equator on the ecliptic plane, the density of the Earth, and the ratio between the lunar attraction and the solar attraction in order to obtain that the final result is equal to the 50” known in its age (today the accepted value is 50. 4”)[18]. Certainly we cannot exclude a priori that, because of the equatorial bulge, the □point moves towards the Sun covering an unknown distance, as well as we cannot deny that the Sun is pushed towards the same point, because of the gravity of all planets and in particular of Jupiter. The problem is the determination of the subdivision of the relative motions between the abovementioned causes. Unfortunately the theory of the precession is founded on assumptions that are not certain and that, opportunely changed according to various criteria, imply a shift of the □point that is largely variable. A typical example is the work of the famous mathematician Daniel Bernoulli, who, in unsuspicious times, starting from some experimental observations, according to a report by d'Alembert [14], came to the conclusion that the ratio between the forces that the Moon and the Sun exert on the equatorial bulge is 5/2. Leaving unaltered all the other hypotheses of Newton, included

[17] A ring of moons around the Earth will cause a null recoil, as it is easy to verify. Jupiter and Saturn, in opposition, would cause a residual recoil of the Sun equal to 45”.

[18] Newton uses such tricks even for the verification of the law of the inverse square distance, as well as for the speed of sound in the air. And we can imagine that if Newton had known that a part of the movement of the □point could be attributed to the Sun, he would have adjusted everything in order to obtain a perfect result with his calculations.

the solidification of the ring of moons with the terrestrial globe, this implies that only 35" of the total 50" experimentally observed can be justified. If the opinion of Bernoulli had survived [19], Celestial Mechanics would already have serious problems of survival in the ancient 1700 because it would have to justify much larger forward shifts of the perihelia of the various planets.

Thus it is easy to imagine that the parameters relative to the precession were always chosen in such a way to assign the entire shift only to the point γ. On the basis of these remarks, leaving a thorough examination to another paper, we can only conclude that only precise experimental measurements, opportunely set out, can establish with certainty both the absolute motion of the Sun and the γ point, even taking into account the subtle remarks by K. Homann concerning the non-decrease of the precession [10].

9 On the General Relativity Principle

We want to clarify a common way of thinking according to which the truthfulness of Galilei's law on the fall of bodies is directly related to the equality of the inertial mass and the gravitational mass. Actually the reverse is true.

If Galilei's law is not verified then there can be the perfect equality of the inertial and gravitational masses. Even this commonplace can be retraced up to Newton (the independence of the oscillation times of pendulums from the masses used). In fact, if it is true that the inertial mass and the gravitational mass are identical or proportional, then it is true Newton's law for the fall of bodies, i.e.

$$a_m = \frac{GM}{R^2}\left(1+\frac{m}{M}\right) = g\left(1+\frac{m}{M}\right) \tag{64}$$

and thus Galilei's law is not verified. Einstein, whose only purpose was the extension of the validity range of the Special Relativity, made recourse to Galilei's law, a law that no researcher ever cared to verify fully [20], starting from Galilei himself, and that maybe none will care to verify. Actually, according to the historian Di Trocchio [12, pag. 14], in 1978 two researchers, C.B. Adler and B. Coulter decided to replicate Galilei's experiment (from the Tower of Pisa) and verified that the two balls reach the Earth with a very small delay. This delay does not justify completely the hypothesis of Aristotle according to which a body having double weight should reach the Earth with double speed, but however the heavier body falls first. Always Di Trocchio states that the Aristotelians could modify their theory because of this result. In this context we must observe that Adler and Coulter did not take into account the possible revision of the said experiment according to the consequences of the theory of Newton [21]. It is interesting to note that, given (64), the above-mentioned researchers should have appreciated no difference with the naked eye, given the extreme smallness of the corrective term. If this difference is even perceivable without special devices, according to Newton, we must take into account other causes like, for example, the non-homogeneity of the subsoil (concentrated masses) that amplify the phenomenon. In any case, if we accept Newton's theory, we must say that the Aristotelians were fully right when they stated that a body with double weight reach the Earth with double speed, since the mentioned very small speeds

[19] It is clear that at the time none suspected the existence of the real shift of the Sun that any gravitational theory allows to calculate, as it is easy to verify.

[20] There were some unofficial tests that appear to have established that the heaviest body falls first [12].

[21] Note that, in many fundamental points, the theory of Galilei is not reread and reinterpreted by the following theory of Newton, but superimpose and survives to the latter, notwithstanding the existence of strong contradictions. An example is just the fall law, while the other concerns relative motions. The last problem is temporarily set apart even because at present both the gravity and the electric field seem to be exclusively positional fields -another hypothesis undemonstrated but considered certain and expected.

sum up and are masked by the common identical speed, much greater, that the Earth gives in an equal way to all the bodies. In fact, given (64), while the acceleration g is the (Galileian) acceleration that the Earth exerts on all the falling bodies in the same way, the acceleration (gm/M) is the one that the generic falling body exerts on the Earth as a function of its own different mass.
Einstein, before the ultimate formulation of General Relativity, using the famous gedanken experiment of the lift in a gravitational field, deduced brilliantly, on the basis of the identity principle of inertia and gravitation, or better according to Galilei's law, the deviation of a light ray caused by a celestial body. But maybe, if we make a deepest analysis, since any photon has its own mass $h\nu/C^2$, a ray of light in a gravitational field, besides verifying Einstein's relation for that part of a_m that we can briefly call the Galileian part, should also exhibit the same behaviour that it shows throughout a prism, just because perhaps the gravitational masses measure with extreme precision the smallest differences of the weight, due to the part represented by (gm/M). And also the clocks should measure different times if they have different masses. Thus there is not one single space-time, but many, each being a function of the secondary mass. Concerning the forward shift of Mercury perihelion a clarification is needed. According to General Relativity, a great gravitational mass like the Sun changes the geometry of space-time, so that an infinitesimal orbiting mass undergoes a real forward shift of the perihelion[22]. In the Newtonian language, this means that Newton's force is accompanied by an additional force that is proportional to the fourth power of d. According to the present version, this would be instead an apparent forward shift of Mercury's perihelion due to a real backward shift of the Sun, that should be attributed to the active gravitational mass of Mercury. If, even here, we used the language of non-Euclidean geometry, we could say that the curvature of the three-dimensional physical space around a mass is an exclusive function of the peripheral mass. Only when this mass is infinitesimal we would recover the traditional geometry (Lobacewsky). But it seems sufficiently evident that, whatever the gravity law would be, in the transformation from an inertial reference frame to another reference frame anchored to the central mass, we must take into account the remarks of Section 3. On the other hand we must admit that the observing window of Kepler, from which Newton deduced the law of gravitation, is only a very modest neighbourhood to explore, if it is compared, for example, with the radius of a galaxy. So the pretence of an extension of a law deduced in such a restricted context could be very hazardous; it is sufficient to think to the inner singularity of Newton's law (and Coulomb's law) for masses (or charges) when their distance goes to zero.

10 Conclusions

Certainly, due to the extension of the above-mentioned subjects, the present paper cannot be exhaustive. The lack of further analytical developments that take into account the mobility of the central mass forces this paper to be only the statement of a fundamental problem, that arises from the unaware correction made by Newton to the experimental (?) law of Galileo Galilei. On the other hand, it is well-known that the two-body problem is the basis of any perturbative theory, so one of its variants is a source of further analytical complications that must be overcome if we want to establish the possible internal consistence of the searched correction. The fact that the astronomers have always thought of a Sun absolutely fixed in the space, with all the relevant and consequential mistakes, contrary to the vain Newtonian assumption that the secondary mass, because of its gravity, moves the central mass, is more and more demonstrated by the extreme fact that one of the gravitational perturbative theories distributes on the entire Keplerian orbit the whole mass of the planet, nullifying completely the recoil of the central mass.
If we accept the validity of relation (64), we can understand that if Galilei really had executed the

[22] But if, in the context of General Relativity, we also consider the mass of Mercury, given its smallness, the results should not be much different from the Newtonian results and then we would have 43" for the relativistic forward shift plus 44" for the recoil!

famous experiment of the Tower of Pisa, even with modern devices, he would have never ascertained the infinitesimal difference given by the said formula. So, maybe, a precise experiment executed with that goal could give surprising results. Indeed, if it is true that the heaviest body falls first [12], this is even more surprising and conflicts with the extreme smallness of the difference that should be detected according to (64). In fact, unless the excessive closeness to the Earth surface would imply radical modifications, Galilei's law should be true only if the extreme smallness of the difference to detect would be true. But it is clear that, admitting the validity of Newton's law, the situation is completely different at the solar system scale, where the masses can be considered like points. In such a context, and mostly in the case of the Earth{Moon and Sun{Jupiter pairs, the m/M ratio is not negligible and gives rise to an undeniable shift of the primary mass, contrary to any statement by Newton in primis and by the mathematicians and astronomers Laplace, Clariaut, Euler, Tisserand, Newcomb, etc., until today.

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11 Notes about the Author

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